

S-duality and 2d Topological QFT

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ABSTRACT:

We study the superconformal index for the class of $\mathcal{N} = 2$ 4d superconformal field theories recently introduced by Gaiotto [1]. These theories are defined by compactifying the $(2, 0)$ 6d theory on a Riemann surface with punctures. We interpret the index of the 4d theory associated to an n -punctured Riemann surface as the n -point correlation function of a 2d topological QFT living on the surface. Invariance of the index under generalized S-duality transformations (the mapping class group of the Riemann surface) translates into associativity of the operator algebra of the 2d TQFT. In the A_1 case, for which the 4d SCFTs have a Lagrangian realization, the structure constants and metric of the 2d TQFT can be calculated explicitly in terms of elliptic gamma functions. Associativity then holds thanks to a remarkable symmetry of an elliptic hypergeometric beta integral, proved very recently by van de Bult [2].

KEYWORDS: CFT, S-duality, TQFT.

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1. Introduction

Electric-magnetic duality (S-duality) in four-dimensional gauge theory has a deep connection with two-dimensional modular invariance. The canonical example is the $SL(2, \mathbb{Z})$ symmetry of $\mathcal{N} = 4$ super-Yang-Mills, which can be interpreted as the modular group of a torus. A physical picture for this correspondence is provided by the existence of the six-dimensional $(2, 0)$ superconformal field theory, whose compactification on a torus of modular parameter τ yields $\mathcal{N} = 4$ SYM with holomorphic coupling τ (see [3] for a recent discussion).

Gaiotto [1] has recently discovered a beautiful generalization of this construction. A large class of $\mathcal{N} = 2$ superconformal field theories in 4d is obtained by compactifying a twisted version of the $(2, 0)$ theory on a Riemann surface Σ , of genus g and with n punctures. The complex structure moduli space $\mathcal{T}_{g,n}/\Gamma_{g,n}$ of Σ is identified with the space of exactly marginal couplings of the 4d theory. The mapping class group $\Gamma_{g,n}$ acts as the group of generalized S-duality transformations of the 4d theory. A striking correspondence between the Nekrasov's instanton partition function [4] of the 4d theory and Liouville field theory on Σ has been conjectured in [5] and further explored in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Relations to string/M theory have been discussed in [18, 19, 20, 21]. See also [22, 23, 24].

In this paper we study the superconformal index [25] for this class of 4d SCFTs. The index captures “cohomological” information about the protected states of the theory. By construction, it counts (with signs) the protected states of the theory, up to equivalence relations that set to zero all sequences of short multiplets that may in principle recombine into long multiplets.

The index is invariant under continuous deformations of the theory, and is also expected to be invariant under the S-duality group $\Gamma_{g,n}$. *Assuming S-duality*, this implies that the index must be computed by a topological QFT living on Σ . The usual physical arguments involving the $(2,0)$ theory give a “proof” of this assertion, as follows. The index has a path integral representation [25] as the partition function of the 4d theory on $S^3 \times S^1$, twisted by various chemical potentials, which uplifts to a (suitably twisted) path integral of the $(2,0)$ theory on $S^3 \times S^1 \times \Sigma$. This path integral must be independent of the metric on Σ . In the limit of small Σ we recover the 4d definition; in the opposite limit of large Σ we expect a purely 2d description. Each puncture on Σ should be regarded as an operator insertion. By this logic, the index must be equal to the n -point correlation function of some TQFT on Σ . The question is whether one can describe this TQFT more directly, and in the process *check* the S-duality of the index.

It is likely that a “microscopic” Lagrangian formulation of the 2d TQFT may be derived from the dimensional reduction of the twisted $(2,0)$ theory that we have just described, but we will not pursue this here. Our approach will be to start with the 4d definition of the index [25] and write its concrete expression for Gaiotto’s A_1 theories, which have a 4d Lagrangian description. We show in section 2 that the index does indeed take the form expected for a correlator in a 2d TQFT. We then evaluate explicitly the structure constants and metric of the TQFT operator algebra, and check its associativity, which is the 2d counterpart of S-duality (section 3). The metric and structure constants have elegant expressions in terms of elliptic Gamma functions and the index in terms of elliptic Beta integrals, a set of special functions which are a new and active branch of mathematical research, see *e.g.* [26, 27, 28] and references therein. For Gaiotto’s A_1 theories associativity of the topological algebra (and thus S-duality) hinges on the invariance of a special case of the $E^{(5)}$ elliptic Beta integral under the Weyl group of F_4 . A proof of this symmetry appeared on the math ArXiv just as this paper was nearing completion [2].¹ In a related physical context, elliptic identities have been used in [29] (following [30]) to prove equality of the superconformal index for Seiberg-dual pairs of $\mathcal{N} = 1$ gauge theories.

It is also natural to ask how things work for the original paradigm of a theory exhibiting S-duality, namely $\mathcal{N} = 4$ SYM. From the viewpoint of the superconformal index the only non-trivial $\mathcal{N} = 4$ dual pairs are the theories based on $SO(2n+1)/Sp(n)$ gauge groups. We study

¹We are grateful to Fokko J. van de Bult for sending us a draft of [2] prior to publication.

these cases in Appendix A. We write integral expressions for the index of two dual theories and check their equality “experimentally”, for the first few orders in a series expansion in the chemical potentials. It would be nice to find an analytic proof.

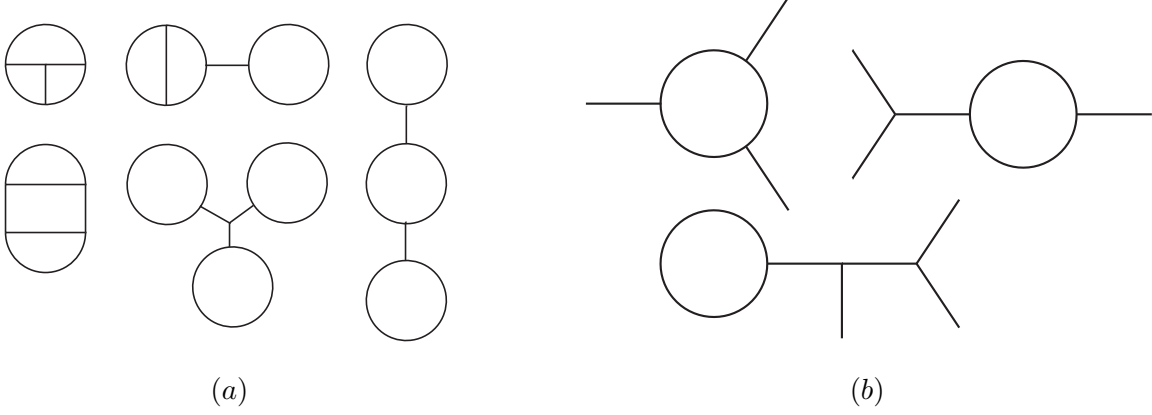


Figure 1: (a) Generalized quiver diagrams representing $\mathcal{N} = 2$ superconformal theories with gauge group $SU(2)^6$ and no flavor symmetries ($N_G = 6$, $N_F = 0$). There are five different theories of this kind. The internal lines of a diagram represent an $SU(2)$ gauge group and the trivalent vertices the trifundamental chiral matter. (b) Generalized quiver diagrams for $N_G = 3$, $N_F = 3$. Each external leg represents an $SU(2)$ flavor group. The upper left diagram corresponds to the $\mathcal{N} = 2$ \mathbb{Z}_3 orbifold of $\mathcal{N} = 4$ SYM with gauge group $SU(2)$.

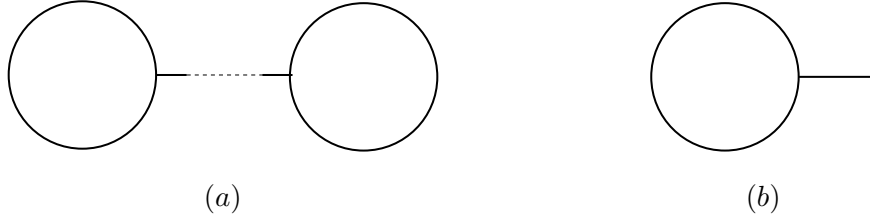


Figure 2: An example of a degeneration of a graph and appearance of flavour punctures. As one of the gauge couplings is taken to zero the corresponding edge becomes very long. Cutting the edge, each of the two resulting semi-infinite open legs will be associated to chiral matter in an $SU(2)$ flavor representation. In this picture setting the coupling of the middle legs in (a) to zero gives two copies of the theory represented in (b), namely an $SU(2)$ gauge theory with a chiral field in the bifundamental representation of the gauge group and in the fundamental of a flavour $SU(2)$.

We end this introduction by recalling the basics of Gaiotto’s analysis [1]. The main achievement of [1] is a purely four-dimensional construction of the SCFT implicitly defined by compactifying the A_{N-1} $(2,0)$ theory on Σ . In the A_1 case an explicit Lagrangian description is available, in terms of a generalized quiver with gauge group $SU(2)^{N_G}$, see Figure 1 for examples. The internal edges of a diagram correspond to the $SU(2)$ gauge groups, the external legs to $SU(2)$ flavor groups and the cubic vertices to chiral fields in the trifundamental representation (fundamental under each of the groups joining at the vertex). The corresponding Riemann surface is immediately pictured by thickening the lines of the graph into tubes – with the external tubes assumed to be infinitely long, so that they can be viewed as punctures. The plumbing parameters τ_i of the tubes are identified with the holomorphic gauge couplings; the degeneration limit when the surface develops a long tube corresponds to the weak coupling limit $\tau \rightarrow +i\infty$ of the corresponding gauge group (Figure 2). The different patterns of degenerations (pair-of-pants decompositions) of a surface Σ of genus g and N_F punctures give rise to the different connected diagrams with N_F external legs ($SU(2)$ flavor groups) and $N_G = N_F + 3(g - 1)$ internal lines ($SU(2)$ gauge groups). Since the mapping class group permutes the diagrams, the associated field theories must be related by generalized S-duality transformations [1].

In the higher A_{N-1} cases the 4d theories are generically described by more complicated quivers that involve new exotic isolated SCFTs as elementary building blocks. While the correspondence between the index and 2d TQFT is general, in this paper we will focus on the A_1 theories, where explicit calculations can be easily performed.

2. 2d TQFT from the Superconformal Index

The superconformal index is defined as [25]

$$\mathcal{I} = \mathcal{I}^{WR} = \text{Tr}(-1)^F t^{2(E+j_2)} y^{2j_1} v^{-(r+R)}, \quad (2.1)$$

where the trace is over the states of the theory on S^3 (in the usual radial quantization). For definiteness we are considering the “right-handed” Witten index \mathcal{I}^{WR} of [25], which computes the cohomology of the supercharge $\bar{\mathcal{Q}}_{2+}$, in notations [31] where the supercharges are denoted as \mathcal{Q}_α^I , $\bar{\mathcal{Q}}_{I\dot{\alpha}}$, $\mathcal{S}_{I\alpha}$, $\bar{\mathcal{S}}_{\dot{\alpha}}^I$, with $I = 1, 2$ $SU(2)_R$ indices and $\alpha = \pm$, $\dot{\alpha} = \pm$ Lorentz indices. (For the class of superconformal theories that we consider, the left-handed and right-handed Witten indices are equal.) The chemical potentials t , y , and v keep track of various combinations of quantum numbers associated to the superconformal algebra $SU(2, 2|2)$: E is the conformal dimension, (j_1, j_2) the $SU(2)_1 \times SU(2)_2$ Lorentz spins, and (R, r) the quantum numbers under the $SU(2)_R \times U(1)_r$ R-symmetry.²

²Our normalization for the R-symmetry charges is as in [31] and differs from [25]: $R_{\text{here}} = R_{\text{there}}/2$, $r_{\text{here}} = r_{\text{there}}/2$.

For the A_1 generalized quivers the index can be explicitly computed as a matrix integral,

$$\mathcal{I} = \int \prod_{\ell \in \mathcal{G}} [dU_\ell] \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \left[\sum_{i \in \mathcal{G}} f_n \cdot \chi_{adj}(U_i^n) + \sum_{(i,j,k) \in \mathcal{V}} g_n \cdot \chi_{3f}(U_i^n, U_j^n, U_k^n) \right] \right). \quad (2.2)$$

Here $f_n = f(t^n, y^n, v^n)$ and $g_n = g(t^n, y^n, v^n)$, with $f(t, y, v)$ and $g(t, y, v)$ the “single-letter partition functions” for respectively the adjoint and trifundamental degrees of freedom, multiplying the corresponding $SU(2)$ characters. The explicit expressions for f and g will be given in the next section. The $\{U_i\}$ are $SU(2)$ matrices. Their index i run over the $N_G + N_F$ edges of the diagram, both internal (“Gauge”) and external (“Flavor”). The set \mathcal{G} is the set of N_G internal edges while the set \mathcal{V} is the set of trivalent vertices, each vertex being labelled by the triple (i, j, k) of incident edges. The integral over $\{U_\ell, \ell \in \mathcal{P}\}$, with $[dU]$ being the Haar measure, enforces the gauge-singlet condition. All in all, the index \mathcal{I} depends on the chemical potentials t, y, v (through f and g) and on (the eigenvalues of) the N_F unintegrated flavor matrices.

The characters depend on a single angular variable α_i for each $SU(2)$ group U_i . Writing

$$U_i = V_i^\dagger \begin{pmatrix} e^{i\alpha_i} & 0 \\ 0 & e^{-i\alpha_i} \end{pmatrix} V_i, \quad (2.3)$$

we have

$$\chi_{adj}(U_i) = \text{Tr} U_i \text{Tr} U_i - 1 = e^{2i\alpha_i} + e^{-2i\alpha_i} + 1 \equiv \chi_{adj}(\alpha_i), \quad (2.4)$$

$$\begin{aligned} \chi_{3f}(U_i, U_j, U_k) &= \text{Tr} U_i \text{Tr} U_j \text{Tr} U_k = (e^{i\alpha_i} + e^{-i\alpha_i})(e^{i\alpha_j} + e^{-i\alpha_j})(e^{i\alpha_k} + e^{-i\alpha_k}) \\ &\equiv \chi_{3f}(\alpha_i, \alpha_j, \alpha_k), \end{aligned} \quad (2.5)$$

where we have used the fact that $2 \sim \bar{2}$. Integrating over V_i , the Haar measure simplifies to

$$\int [dU_i] = \frac{1}{\pi} \int_0^{2\pi} d\alpha_i \sin^2 \alpha_i \equiv \int d\alpha_i \Delta(\alpha_i). \quad (2.6)$$

We now define

$$\begin{aligned} C_{\alpha_i \alpha_j \alpha_k} &\equiv \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} g_n \cdot \chi_{3f}(n\alpha_i, n\alpha_j, n\alpha_k) \right), \\ \eta^{\alpha_i \alpha_j} &\equiv \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} f_n \cdot \chi_{adj}(n\alpha_i) \right) \hat{\delta}(\alpha_i, \alpha_j) \equiv \eta^{\alpha_i} \hat{\delta}(\alpha_i, \alpha_j), \end{aligned} \quad (2.7)$$

where $\hat{\delta}(\alpha, \beta) \equiv \Delta^{-1}(\alpha) \delta(\alpha - \beta)$ (with the understanding that α and β are defined modulo 2π) is the delta-function with respect to the measure (2.6). Further define the “contraction” of an upper and a lower α labels as

$$A^{\cdots \alpha \cdots} B_{\cdots \alpha \cdots} \equiv \int_0^{2\pi} d\alpha \Delta(\alpha) A^{\cdots \alpha \cdots} B_{\cdots \alpha \cdots}. \quad (2.8)$$

The superconformal index (2.2) can then be suggestively written as

$$\mathcal{I} = \prod_{\{i,j,k\} \in \mathcal{V}} C_{\alpha_i \alpha_j \alpha_k} \prod_{\{m,n\} \in \mathcal{G}} \eta^{\alpha_m \alpha_n}. \quad (2.9)$$

The internal labels $\{\alpha_i, i \in \mathcal{G}\}$ associate to the gauge groups are contracted, while the N_F external labels associated to the flavor groups are left open. The expression (2.9) is naturally interpreted as an N_F -point “correlation function” $\langle \alpha_1 \dots \alpha_{N_F} \rangle_{\mathbf{g}}$, evaluated by regarding the generalized quiver as a “Feynman diagram”. The Feynman rules assign to each trivalent vertex the cubic coupling $C_{\alpha\beta\gamma}$, and to each internal propagator the inverse metric $\eta^{\alpha\beta}$. S-duality implies that the superconformal indices calculated from two diagrams with the same (N_F, N_G) must be equal. These properties can be summarized in the statement that the superconformal index is evaluated by a 2d Topological QFT (TQFT).

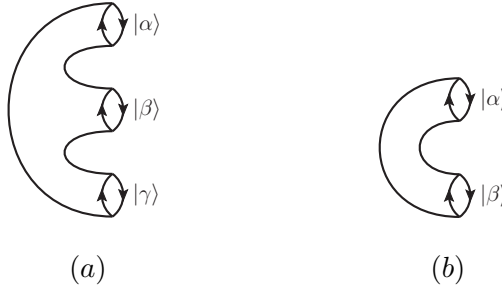


Figure 3: (a) Topological interpretation of the structure constants $C_{\alpha\beta\gamma} \equiv \langle C | |\alpha\rangle |\beta\rangle |\gamma\rangle$. The path integral over the sphere with three boundaries defines $\langle C | \in \mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{H}^*$. (b) Analogous interpretation of the metric $\eta_{\alpha\beta} \equiv \langle \eta | |\alpha\rangle |\beta\rangle$, with $\langle \eta | \in \mathcal{H}^* \otimes \mathcal{H}^*$, in terms of the sphere with two boundaries.

At the informal level sufficient for our discussion, a 2d TQFT [32, 33] can be characterized in terms of the following data: a space of states \mathcal{H} ; a non-degenerate, symmetric metric $\eta: \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{C}$; and a completely symmetric triple product $C: \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \rightarrow \mathbb{C}$. The states in \mathcal{H} are understood physically as wavefunctionals of field configurations on the “spatial” manifold S^1 . The metric and triple product are evaluated by the path integral over field configurations on the sphere with respectively two and three boundaries (Figure 3). The 2d surfaces where the TQFT is defined are assumed to be oriented, so the S^1 boundaries inherit a canonical orientation. To a boundary of inverse orientation (with respect to the canonical one) is associated the *dual* space \mathcal{H}^* . Choosing a basis for \mathcal{H} , we can specify the metric and triple product in terms of $\eta_{\alpha\beta} \equiv \eta(|\alpha\rangle, |\beta\rangle)$ and $C_{\alpha\beta\gamma} \equiv C(|\alpha\rangle, |\beta\rangle, |\gamma\rangle)$, or

$$\eta = \sum_{\alpha, \beta} \eta_{\alpha\beta} \langle \alpha | \langle \beta |, \quad C = \sum_{\alpha, \beta, \gamma} C_{\alpha\beta\gamma} \langle \alpha | \langle \beta | \langle \gamma |. \quad (2.10)$$

The inverse metric $\eta^{\alpha\beta}$ is associated to the sphere with two boundaries of inverse orientation, and as its name suggests it obeys $\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta_{\gamma}^{\alpha}$, see Figure 4. Index contraction corresponds geometrically to gluing of S^1 boundary of compatible orientation.

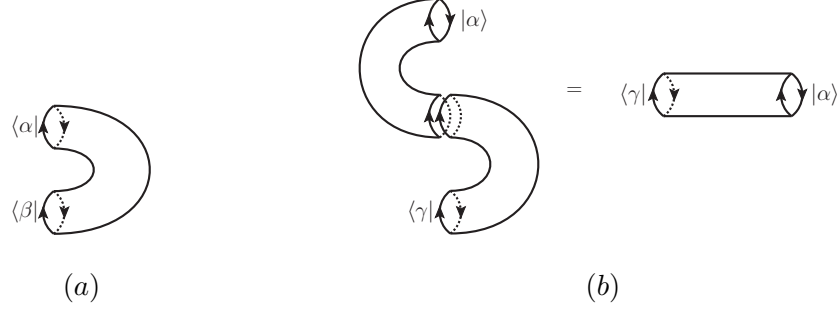


Figure 4: Topological interpretation of **(a)** the inverse metric $\eta^{\alpha\beta}$, **(b)** the relation $\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta_{\alpha}^{\gamma}$. By convention, we draw the boundaries associated with upper indices facing left and the boundaries associated with the lower indices facing right.

The metric and triple product obey natural compatibility axioms which can be simply summarized by the statement that the metric and its inverse are used to lower and raise indices in the usual fashion. Finally the crucial requirement: the structure constants $C_{\alpha\beta}^{\gamma} \equiv C_{\alpha\beta\epsilon}\eta^{\epsilon\gamma}$ define an associative algebra

$$C_{\alpha\beta}^{\delta} C_{\delta\gamma}^{\epsilon} = C_{\beta\gamma}^{\delta} C_{\delta\alpha}^{\epsilon}, \quad (2.11)$$

as illustrated in Figure 5. From these data, arbitrary n -point correlators on a genus g surface can be evaluated by factorization (= pair-of-pants decomposition of the surface). The result is guaranteed to be independent of the specific decomposition.

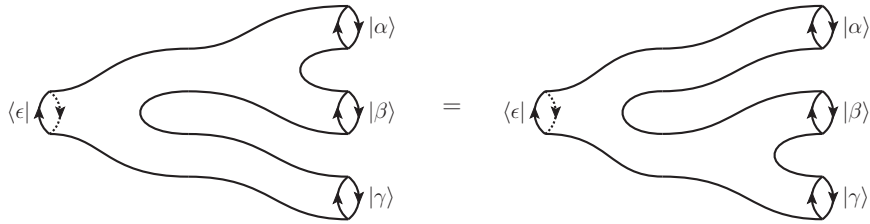


Figure 5: Pictorial rendering of the associativity of the algebra.

In our case the space \mathcal{H} is spanned by the states $\{|\alpha\rangle, \alpha \in [0, 2\pi)\}$, where α parametrizes the $SU(2)$ eigenvalues, equ.(2.3). Alternatively we may “Fourier transform” to the basis of irreducible $SU(2)$ representations, $\{|R_K\rangle, K \in \mathbb{Z}_+\}$, see Appendix B. We have concrete expressions (2.7, 2.8) for the cubic couplings $C_{\alpha\beta\gamma}$ and for the *inverse* metric $\eta^{\alpha\beta}$, which are

Letters	E	j_1	j_2	R	r	\mathcal{I}
ϕ	1	0	0	0	-1	$t^2 v$
λ_{\pm}^1	$\frac{3}{2}$	$\pm\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-t^3 y, -t^3 y^{-1}$
$\bar{\lambda}_{2+}$	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-t^4/v$
\bar{F}_{++}	2	0	1	0	0	t^6
$\partial_{-+}\lambda_+^1 + \partial_{++}\lambda_-^1 = 0$	$\frac{5}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	t^6
q	1	0	0	$\frac{1}{2}$	0	t^2/\sqrt{v}
$\bar{\psi}_+$	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-t^4\sqrt{v}$
$\partial_{\pm+}$	1	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0	$t^3 y, t^3 y^{-1}$

Table 1: Contributions to the index from “single letters”. We denote by $(\phi, \bar{\phi}, \lambda_{\alpha}^I, \lambda_{I\dot{\alpha}}, F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}})$ the components of the adjoint $\mathcal{N} = 2$ vector multiplet, by $(q, \bar{q}, \psi_{\alpha}, \bar{\psi}_{\dot{\alpha}})$ the components of the trifundamental $\mathcal{N} = 1$ chiral multiplet, and by $\partial_{\alpha\dot{\alpha}}$ the spacetime derivatives. Here $I = 1, 2$ are $SU(2)_R$ indices and $\alpha = \pm, \dot{\alpha} = \pm$ Lorentz indices.

manifestly symmetric under permutations of the indices. Formal inversion of (2.8) gives the metric $\eta_{\alpha\beta} \equiv (\eta^{\alpha})^{-1}\hat{\delta}(\alpha, \beta)$. Finally with the help of (2.8) we can raise, lower and contract indices at will. On physical grounds we expect these formal manipulations to make sense, since the superconformal index is well-defined as a series expansion in the chemical potential t , which should have a finite radius of convergence [25]. The explicit analysis of sections 3 and 4 will confirm these expectations. We will find expressions for the index as analytic functions of the chemical potentials. Our definitions satisfy the axioms of a 2d TQFT by construction, and independently of the specific form of the functions $f(t, y, v)$ and $g(t, y, v)$, except for the associativity axiom, which is completely non-trivial. Associativity of the 2d topological algebra is equivalent to 4d S-duality, and it can only hold for very special choices of field content, encoded in the single-letter partition functions f and g .

3. Associativity of the Algebra

In this section we determine explicitly the structure constants and the metric of the TQFT and write them in terms of elliptic Beta integrals. With the help of a recent mathematical result [2] we prove analytically the associativity of the topological algebra.

3.1 Explicit Evaluation of the Index

The “single letters” contributing to the index, which must obey $\bar{\Delta} \equiv E - 2j_2 - 2R + r = 0$ [25], are enumerated in Table 1. The first block of the Table shows the contributing letters from

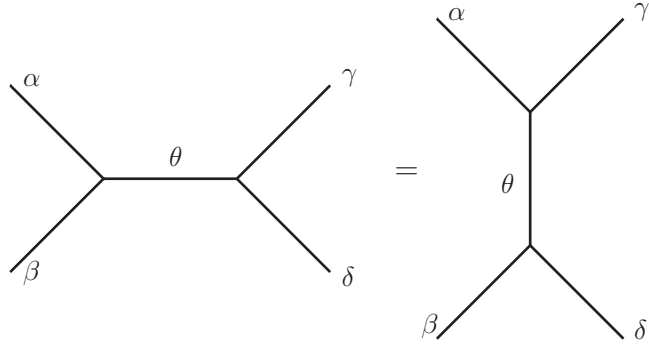


Figure 6: The basic S-duality channel-crossing. The two diagrams are two equivalent (S-dual) ways to represent the $\mathcal{N} = 2$ gauge theory with a single gauge group $SU(2)$ and four $SU(2)$ flavour groups, which is the basic building block of the A_1 generalized quiver theories. The indices on the edges label the eigenvalues of the corresponding $SU(2)$ groups.

the adjoint $\mathcal{N} = 2$ vector multiplet (associated to each internal edge of a graph), including the equations of motion constraint. The second block shows the contributions from the $\mathcal{N} = 1$ chiral multiples in the trifundamental representation, associated to each cubic vertex. Finally the last line of the Table shows the spacetime derivatives contributing to the index. Since each field can be hit by an arbitrary number of derivatives, the derivatives give a multiplicative contribution to the single-letter partition functions of the form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (t^3 y)^m (t^3 y^{-1})^n = \frac{1}{(1 - t^3 y)(1 - t^3 y^{-1})}. \quad (3.1)$$

All in all, the single letter partition function are given by

$$\text{adjoint} \quad : \quad f(t, y, v) = \frac{t^2 v - \frac{t^4}{v} - t^3(y + y^{-1}) + 2t^6}{(1 - t^3 y)(1 - t^3 y^{-1})}, \quad (3.2)$$

$$\text{trifundamental} \quad : \quad g(t, y, v) = \frac{\frac{t^2}{\sqrt{v}} - t^4 \sqrt{v}}{(1 - t^3 y)(1 - t^3 y^{-1})}. \quad (3.3)$$

We are now ready to check explicitly the basic S-duality move – S-duality with respect to one of the $SU(2)$ gauge groups, represented graphically as channel-crossing with respect to one of the edges of the graph (Figure 6). The full S-duality group of a graph is generated by repeated applications of the basic move to different edges. The contribution to the index

from the left graph in Figure 6 is

$$\mathcal{I} = \int d\theta \Delta(\theta) \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} [f_n \cdot \chi_{adj}(n\theta) + g_n \cdot \chi_{3f}(n\alpha, n\beta, n\theta) + g_n \cdot \chi_{3f}(n\theta, n\gamma, n\delta)] \right). \quad (3.4)$$

Substituting the expressions for the characters,

$$\mathcal{I} = \frac{e^{\sum_{n=1}^{\infty} \frac{f_n}{n}}}{\pi} \int_0^{2\pi} d\theta \sin^2 \theta e^{\sum_{n=1}^{\infty} \frac{2f_n}{n} \cos 2n\theta} e^{\sum_{n=1}^{\infty} \frac{8g_n}{n} [\cos n\alpha \cos n\beta + \cos n\gamma \cos n\delta] \cos n\theta}, \quad (3.5)$$

where $f_n \equiv f(t^n, y^n, v^n)$ and $g_n \equiv f(t^n, y^n, v^n)$. S-duality of the index is the statement this integral is invariant under permutations of the external labels $\alpha, \beta, \gamma, \delta$. Since symmetries under $\alpha \leftrightarrow \beta$ and (independently) under $\gamma \leftrightarrow \delta$ are manifest, the non-trivial requirement is symmetry under $\beta \leftrightarrow \gamma$, which gives the index associated to the crossed graph on the right of Figure 6.

The integrand of (3.5) is *not* invariant under $\beta \leftrightarrow \gamma$, but the integral is, as once can check order by order in a series expansion in the chemical potential t . Here is how things work to the first non-trivial order. We expand the integrand in t around $t = 0$, and set $y = v = 1$ for simplicity. The single-letter partition functions behave as

$$f(t, y = 1, v = 1) \sim t^2 - 2t^3, \quad g(t, y = 1, v = 1) \sim t^2 - t^4. \quad (3.6)$$

The first non-trivial check is for the coefficient of \mathcal{I} of order $O(t^4)$,

$$\mathcal{I} \sim t^4 \int_0^{2\pi} d\theta \sin^2 \theta \left(\cos 4\theta + 2 \cos^2 2\theta + 4A_2 \cos 2\theta + 32A_1^2 \cos^2 \theta - \right. \quad (3.7) \\ \left. - 2 \cos 2\theta + 16A_1 \cos \theta \cos 2\theta - 8A_1 \cos \theta \right),$$

where $A_n \equiv \cos n\alpha \cos n\beta + \cos n\gamma \cos n\delta$. Performing the elementary integrals,

$$\mathcal{I} \sim t^4 [6\pi + 2\pi (\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \cos 2\delta + 8 \cos \alpha \cos \beta \cos \gamma \cos \delta)], \quad (3.8)$$

which is indeed symmetric under $\alpha \leftrightarrow \beta \leftrightarrow \gamma \leftrightarrow \delta$. We stress that crossing symmetry depends crucially on the specific form of the single-letter partition functions (3.2) and thus on the specific field content. We have performed systematic checks by calculating the series expansion to several higher orders using *Mathematica*. Fortunately it is possible to give an analytic proof of crossing symmetry of the index, as we now describe.

3.2 Elliptic Beta Integrals and S-duality

The fundamental integral (3.5) can be recast in an elegant way in terms of special functions known as elliptic Beta integrals. We start by recalling the definition of the elliptic Gamma

function, a two parameter generalization of the Gamma function,

$$\Gamma(z; p, q) \equiv \prod_{j,k \geq 0} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}. \quad (3.9)$$

For reviews of the elliptic Gamma function and of elliptic hypergeometric mathematics the reader can consult [26, 27, 28]. Throughout this paper we will use the standard condensed notations

$$\begin{aligned} \Gamma(z_1, \dots, z_k; p, q) &\equiv \prod_{j=1}^k \Gamma(z_j; p, q), \\ \Gamma(z^{\pm 1}; p, q) &= \Gamma(z; p, q) \Gamma(1/z; p, q). \end{aligned} \quad (3.10)$$

Two identities satisfied by the elliptic Gamma function that will be useful to us are

$$\Gamma(z^2; p, q) = \Gamma(\pm z, \pm \sqrt{q} z, \pm \sqrt{p} z, \pm \sqrt{pq} z; p, q), \quad (3.11)$$

$$\Gamma(pq/z; p, q) \Gamma(z; p, q) = 1. \quad (3.12)$$

(As an illustration of the shorthand (3.10), the *rhs* of (3.11) is a product of eight Gamma functions.) Using the definition (3.9), it is straightforward to show [29]

$$\begin{aligned} \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{t^{2n} z^n - t^{4n} z^{-n}}{(1 - t^{3n} y^n)(1 - t^{3n} y^{-n})} \right) &= \Gamma(t^2 z; p, q), \\ \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{2t^{6n} - t^{3n}(y^n + y^{-n})}{(1 - t^{3n} y^n)(1 - t^{3n} y^{-n})} (z^n + z^{-n}) \right) &= -\frac{z}{(1-z)^2} \frac{1}{\Gamma(z^{\pm 1}; p, q)}, \end{aligned} \quad (3.13)$$

where

$$p = t^3 y, \quad q = t^3 y^{-1}. \quad (3.14)$$

With these preparations, the building blocks (2.7) for the index can be written in the following compact form

$$\begin{aligned} C_{\alpha_i \alpha_j \alpha_k} &= \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} g_n \chi_{3f}(n\alpha_i, n\alpha_j, n\alpha_k) \right) = \Gamma \left(\frac{t^2}{\sqrt{v}} a_i^{\pm 1} a_j^{\pm 1} a_k^{\pm 1}; p, q \right), \\ \eta^{\alpha_i} &= \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} f_n \chi_{adj}(n\alpha_i) \right) = \frac{1}{\Delta(\alpha_i)} \frac{(p; p)(q; q)}{4\pi} \Gamma(t^2 v; p, q) \frac{\Gamma(t^2 v a_i^{\pm 2}; p, q)}{\Gamma(a_i^{\pm 2}; p, q)}. \end{aligned} \quad (3.15)$$

Here we have defined $a_i = \exp(i\alpha_i)$ and used

$$\exp \left(\sum_{n=1}^{\infty} \frac{1}{n} f_n \right) = (p; p)(q; q) \Gamma(t^2 v; p, q), \quad (a; b) \equiv \prod_{k=0}^{\infty} (1 - a b^k). \quad (3.16)$$

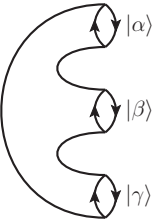


Symbol	Surface	Value
$C_{\alpha\beta\gamma}$		$\Gamma(\frac{t^2}{\sqrt{v}}a^{\pm 1}b^{\pm 1}c^{\pm 1})$
$C_{\alpha\beta}^{\gamma}$		$\frac{i\kappa}{\Delta(\gamma)} \Gamma(t^2 v) \frac{\Gamma(t^2 v c^{\pm 2})}{\Gamma(c^{\pm 2})} \Gamma(\frac{t^2}{\sqrt{v}}a^{\pm 1}b^{\pm 1}c^{\pm 1})$
$\eta^{\alpha\beta}$		$\frac{i\kappa}{\Delta(\alpha)} \Gamma(t^2 v) \frac{\Gamma(t^2 v a^{\pm 2})}{\Gamma(a^{\pm 2})} \hat{\delta}(\alpha, \beta)$

Table 2: The structure constants and the metric in terms of elliptic Gamma functions. For brevity we have left implicit the parameters of the Gamma functions, $p = t^3 y$ and $q = t^3 y^{-1}$. We have defined $a \equiv \exp(i\alpha)$, $b \equiv \exp(i\beta)$, and $c \equiv \exp(i\gamma)$. Recall also $\kappa \equiv (p; p)(q; q)/4\pi i$ and $\Delta(\alpha) \equiv (\sin^2 \alpha)/\pi$.

Again, the reader should keep in mind that the *rhs* of the first line in (3.15) is a product of eight elliptic Gamma functions according to the condensed notation (3.10).

Collecting all these definitions the fundamental integral (3.5) becomes

$$\kappa \Gamma(t^2 v; p, q) \oint \frac{dz}{z} \frac{\Gamma(t^2 v z^{\pm 2}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \Gamma(\frac{t^2}{\sqrt{v}}a^{\pm 1}b^{\pm 1}z^{\pm 1}; p, q) \Gamma(\frac{t^2}{\sqrt{v}}c^{\pm 1}d^{\pm 1}z^{\pm 1}; p, q), \quad pq = t^6, \quad (3.17)$$

with $\kappa \equiv (p; p)(q; q)/4\pi i$. As it turns out, this integral fits into a class of integrals which are an active subject of mathematical research, the elliptic Beta integrals

$$E^{(m)}(t_1, \dots, t_{2m+6}) \sim \oint \frac{dz}{z} \frac{\Gamma(t_1 z, \dots, t_{2m+6} z; p, q)}{\Gamma(z^{\pm 2}; p, q)}, \quad \prod_{k=1}^{2m+6} t_k = (pq)^{m+1}. \quad (3.18)$$

Our integral is a special case of $E^{(5)}$. Elliptic Beta integrals have very interesting symmetry properties. For instance the symmetry of $E^{(2)}$ is related to the Weyl group of E_7 . Very

recently van de Bult proved [2] that special cases of the $E^{(5)}$ integral, which are equivalent to (3.17), are invariant under the Weyl group of F_4 . In particular (3.17) is invariant under $b \leftrightarrow c$. This is theorem 3.2 in [2], with the parameters $\{t_{1,2,3,4}, b\}$ of [2] related to the parameters $\{a, b, c, d, t^2 v\}$ in our equation (3.17) by the substitution

$$t_1 \rightarrow \frac{t^2}{\sqrt{v}} a b, \quad t_2 \rightarrow \frac{t^2}{\sqrt{v}} a/b, \quad t_3 \rightarrow \frac{t^2}{\sqrt{v}} c d, \quad t_4 \rightarrow \frac{t^2}{\sqrt{v}} c/d, \quad b \rightarrow t^2 v. \quad (3.19)$$

This completes the proof of crossing symmetry of the fundamental integral (3.5).



Figure 7: Handle-creating operator \mathcal{J}_α

The expressions for the structure constants and metric of the topological algebra in terms of the elliptic Gamma functions are summarized in Table 2. These expressions are analytic functions of their arguments, except for the metric $\eta^{\alpha\beta}$ which contains a delta-function. One can try and use the results of the theory of elliptic Beta integrals to represent the delta-function in a more elegant way, indeed such a representation is sometimes available in terms of a contour integral [34]. However, for generic choices of the parameters, the definition of [34] involves contour integrals not around the unit circle and thus using this representation one presumably should also change the prescription (2.8) for contracting indices. In the limit $v \rightarrow t$ the relevant contours do approach the unit circle and the formalism of [34] yields elegant expressions. This limit is however slightly singular. We discuss it in Appendix C.

As a simple illustration of the use of the expressions in Table 2 let us compute the superconformal index of the theory associated to diagram (b) in Figure 2. This is essentially the “handle-creating” vertex \mathcal{J}_α of the TQFT, Figure 7. We have

$$\mathcal{J}_\alpha = C_{\alpha\beta\gamma} \eta^{\beta\gamma} = \kappa \Gamma(t^2 v) \Gamma\left(\frac{t^2}{\sqrt{v}} a^{\pm 1}\right)^2 \oint \frac{dz}{z} \frac{\Gamma(t^2 v z^{\pm 2})}{\Gamma(z^{\pm 2})} \Gamma\left(\frac{t^2}{\sqrt{v}} z^{\pm 2} a^{\pm 1}\right). \quad (3.20)$$

Multivariate extensions of elliptic Beta integrals have appeared in the calculation of the superconformal index for pairs of $\mathcal{N} = 1$ theories related by Seiberg duality [29]. Unlike our $\mathcal{N} = 2$ superconformal cases, there is no continuous deformation relating two Seiberg-dual theories, and it is not a priori obvious that their indices, evaluated at the free UV fixed points, should coincide – but it turns out that they do, thanks to identities satisfied by these multivariate integrals [35]. See also [36]. In Appendix A we tackle the $\mathcal{N} = 4$ case, evaluating the indices the S-dual pairs with gauge groups $Sp(n)$ and $SO(2n+1)$. Again S-duality predicts

some new identities of elliptic Beta integrals, which we confirm to the first few orders in the t expansion. It appears that there is a general connection between elliptic hypergeometric mathematics and electric-magnetic duality of the index of $4d$ gauge theories.

4. Discussion

A rich class of $4d$ superconformal field theories arise by compactifying the $6d$ $(2,0)$ theory on a punctured Riemann surface Σ [1], and this has inspired a precise dictionary between $4d$ and $2d$ quantities [5, 6, 7, 8, 9]. In this paper we have added a new entry to this dictionary. Previous work has focussed on the relation between the $4d$ theory on S^4 (or more generally on the theory in the Ω background) and Liouville theory on Σ . Here we have considered instead the superconformal index [25], which can be viewed as the partition function of the $4d$ theory on $S^3 \times S^1$, with twisted boundary conditions labelled by three chemical potentials. We have argued that the superconformal index is evaluated by a topological QFT on Σ . In the A_1 case we have computed explicitly the structure constants of the topological algebra and checked its associativity, using a rather non-trivial piece of contemporary mathematics [2]. Physically this result can be regarded as a precise check that the protected spectrum of operators is the same for the $SU(2)^{N_G}$ theories related by the generalized S-dualities of [1].

There are several interesting directions for future research. It would be illuminating to obtain a Lagrangian description of the $2d$ TQFT from a twisted compactification of the $(2,0)$ theory on $S^3 \times S^1$, and reproduce by that route the structure constants evaluated in this paper. The best known example of a topological field theory with observables labelled by the representations of $SU(2)$ is $2d$ Yang-Mills theory, and it is likely that our theory will turn out to be related to it. There is then the related question of finding how this structure can be embedded in string theory, perhaps along the lines of [20]. Finally our work should be extended to the A_{N-1} theories with $N > 2$. While for these theories a $4d$ Lagrangian description is in general lacking, there are indirect ways to construct them by taking limits of known theories. The mathematical structure of the superconformal index is so rigid that it may be possible to determine it by consistency, using purely $4d$ considerations. Alternatively, the “top-down” approach from compactification of the $(2,0)$ theory is expected to give a uniform answer for all the A_{N-1} theories.

We suspect that we are just scratching the surface of a general connection between elliptic hypergeometric mathematics and S-duality. It is possible to generate new elliptic hypergeometric identities by calculating the superconformal index of S-dual theories. Already the simplest S-dualities (from a physical perspective), such as the $SO(2n+1)/Sp(n)$ dualities in $\mathcal{N} = 4$ SYM, lead to identities that to the best of our knowledge have not appeared in the mathematical literature. One may wonder whether the logic can be reversed, and new S-dualities discovered from known elliptic identities. Elliptic Beta integrals are the most gen-

eral known extensions of the classic Euler Beta integral, and as such they are the natural mathematical objects to appear in the calculation of “crossing-symmetric” physical quantities. It is perhaps not coincidental that the mathematics and the physics of the subject are being developed simultaneously, and we can look forward to a fruitful interplay between the two viewpoints.

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A. S-duality for $\mathcal{N} = 4$ $SO(2n+1)/Sp(n)$ SYM

In this Appendix we compute the superconformal indices for $\mathcal{N} = 4$ SYM with gauge groups $SO(2n+1)$ and $Sp(n)$. Since the SO and Sp theories are related by S-duality, their indices are expected to agree. These are in fact the only non-trivial $\mathcal{N} = 4$ cases from the viewpoint of index calculations. Indeed the index depends on the adjoint representation of the group: the A, D, E, F and G cases are manifestly self-dual, and the only interesting duality is $B \leftrightarrow C$.

The characters of the adjoint representations of for $Sp(n)$ and $SO(2n+1)$ are

$$\begin{aligned}\chi_{Sp(n)}(\{z_i\}) &: \sum_{1 \leq i < j \leq n} \left(z_i z_j + z_i z_j^{-1} + z_j z_i^{-1} + z_i^{-1} z_j^{-1} \right) + \sum_{i=1}^n (z_i^2 + z_i^{-2}) + n, \\ \chi_{SO(2n+1)}(\{z_i\}) &: \sum_{1 \leq i < j \leq n} \left(z_i z_j + z_i z_j^{-1} + z_j z_i^{-1} + z_i^{-1} z_j^{-1} \right) + \sum_{i=1}^n (z_i + z_i^{-1}) + n.\end{aligned}\tag{A.1}$$

Their Haar measures are

$$\begin{aligned}Sp(n) &: \int_{Sp(n)} d\mu(z) f(z) = \frac{(-)^n}{2^n n!} \oint_{\mathbb{T}_n} \prod_{j=1}^n \frac{dz_j}{2\pi i z_j} \prod_{j=1}^n (z_j - z_j^{-1})^2 \Delta(\mathbf{z} + \mathbf{z}^{-1})^2 f(z), \\ SO(2n+1) &: \int_{SO(2n+1)} d\mu(z) f(z) = \frac{(-)^n}{2^n n!} \oint_{\mathbb{T}_n} \prod_{j=1}^n \frac{dz_j}{2\pi i z_j} \prod_{j=1}^n (z_j^{1/2} - z_j^{-1/2})^2 \Delta(\mathbf{z} + \mathbf{z}^{-1})^2 f(z),\end{aligned}\tag{A.2}$$

where \mathbb{T}_n is an n -dimensional torus with unit radii and $\Delta(\mathbf{x})$ the van der Monde determinant

$$\Delta(\mathbf{x}) = \prod_{i < j} (x_i - x_j).\tag{A.3}$$

The single letter partition function is in both cases equal to [25]

$$f(t, y) = \frac{3t^2 - 3t^4 - t^3(y + y^{-1}) + 2t^6}{(1 - t^3 y)(1 - t^3 y^{-1})}, \quad (\text{A.4})$$

where for simplicity we have omitted the chemical potentials of the R-charges – we will restore them in the end. Using the identities (3.13),

$$\begin{aligned} e^{\sum_k \frac{f_k}{k} \chi_{Sp(n)}(\{z_i^k\})} &= \Gamma^{3n}(t^2; p, q) (p; p)^n (q; q)^n \prod_{i < j} \frac{z_j^2}{(1 - z_i z_j)^2 (1 - z_i^{-1} z_j)^2} \frac{1}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \\ &\prod_j \frac{-z_j^2}{(1 - z_j^2)^2} \frac{1}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{i < j} \Gamma(t^2 z_i^{\pm 1} z_j^{\pm 1}; p, q)^3 \prod_j \Gamma(t^2 z_i^{\pm 2}; p, q)^3. \end{aligned} \quad (\text{A.5})$$

Recall the definition (3.15) of the product $(x; y)$. Further, using

$$\begin{aligned} \prod_{i < j} (1 - z_i z_j) (1 - z_i / z_j) (1 - z_j / z_i) (1 - 1 / (z_i z_j)) &= \Delta(\mathbf{z} + \mathbf{z}^{-1})^2, \\ \prod_j (1 - z_j^2) (1 - 1 / z_j^2) &= (-1)^n \prod_j (z_j - 1 / z_j)^2, \end{aligned} \quad (\text{A.6})$$

we obtain

$$\begin{aligned} \int_{Sp(n)} d\mu(z) e^{\sum_k \frac{f_k}{k} \chi_{Sp(n)}(\{z_i\})} &= \\ \frac{\Gamma^{3n}(t^2; p, q)}{2^n n!} (p; p)^n (q; q)^n \oint \prod_j \frac{dz_j}{2\pi i z_j} \prod_{i < j} \frac{\Gamma(t^2 z_i^{\pm 1} z_j^{\pm 1}; p, q)^3}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_j \frac{\Gamma(t^2 z_j^{\pm 2}; p, q)^3}{\Gamma(z_j^{\pm 2}; p, q)}. \end{aligned} \quad (\text{A.7})$$

In complete analogy we obtain for the $SO(2n+1)$ gauge group

$$\begin{aligned} \int_{SO(2n+1)} d\mu(z) e^{\sum_k \frac{f_k}{k} \chi_{SO(2n+1)}} &= \\ \frac{\Gamma^{3n}(t^2; p, q)}{2^n n!} (p; p)^n (q; q)^n \oint \prod_j \frac{dz_j}{2\pi i z_j} \prod_{i < j} \frac{\Gamma(t^2 z_i^{\pm 1} z_j^{\pm 1}; p, q)^3}{\Gamma(z_i^{\pm 1} z_j^{\pm 1}; p, q)} \prod_j \frac{\Gamma(t^2 z_j^{\pm 1}; p, q)^3}{\Gamma(z_j^{\pm 1}; p, q)}. \end{aligned} \quad (\text{A.8})$$

S-duality predicts that the integrals (A.7) and (A.8) must agree. For $Sp(1) \cong SO(3)$ this is trivially checked by a change of variable: in the $SO(3)$ integral make the substitution $z \rightarrow y = \sqrt{z}$. The case of $Sp(2) \cong SO(5)$ is also trivial (as it should be). Define $\hat{z}_1 = \sqrt{z_1 z_2}$ and $\hat{z}_2 = \sqrt{z_1 / z_2}$. Then in (A.8) the first product is exchanged with the second with a doubled power of the z argument and we obtain (A.7). We have checked for the first few orders in a series expansion in t that (A.7) (A.8) also agree for higher rank groups. We do not have an analytic proof of this statement.

Given an orthonormal basis e_i of \mathbb{R}^n the root system of C_n ($Sp(n)$) consists of vectors of the form $X(C_n) = \{\pm 2e_i, \pm e_i \pm e_j, i < j\}$. The root system of B_n ($SO(2n+1)$) on the other

hand consists of vectors of the form $X(B_n) = \{\pm e_i, \pm e_i \pm e_j, i < j\}$. These two systems are dual to one other. The integrands in (A.7) and (A.8) are given by

$$\prod_{\alpha \in X} \frac{\Gamma(t^2 e^\alpha; p, q)^3}{\Gamma(e^\alpha; p, q)}, \quad (\text{A.9})$$

where X is the corresponding root system and we formally identify $z_i = e^{e_i}$. In this language it is easy to understand why the integrals (A.8) with $SO(3)/SO(5)$, (A.7) with $Sp(1)/Sp(2)$ are equal to one other. In these cases the two root systems are linear transformations of one other, *i.e.* rescaling and in the case of $Sp(2)/SO(5)$ also rotation. For higher n the relation is more complicated. For example for $n = 3$ the $SO(7)$ lattice is a cube and the $Sp(3)$ lattice is an octahedron.

Finally, let us indicate how the expressions for the indices are modified by adding the chemical potentials for the R-symmetry charges [25]. The only differences are in the numerators of (A.7, A.8), which become

$$\begin{aligned} Sp(n) : \quad & \prod_{i < j} \Gamma(t^2 v z_i^{\pm 1} z_j^{\pm 1}; p, q) \Gamma\left(\frac{t^2}{w} z_i^{\pm 1} z_j^{\pm 1}; p, q\right) \Gamma\left(\frac{wt^2}{v} z_i^{\pm 1} z_j^{\pm 1}; p, q\right) \quad (\text{A.10}) \\ & \prod_j \Gamma(t^2 v z_j^{\pm 2}; p, q) \Gamma\left(\frac{t^2}{w} z_j^{\pm 2}; p, q\right) \Gamma\left(\frac{wt^2}{v} z_j^{\pm 2}; p, q\right), \\ SO(2n+1) : \quad & \prod_{i < j} \Gamma(t^2 v z_i^{\pm 1} z_j^{\pm 1}; p, q) \Gamma\left(\frac{t^2}{w} z_i^{\pm 1} z_j^{\pm 1}; p, q\right) \Gamma\left(\frac{wt^2}{v} z_i^{\pm 1} z_j^{\pm 1}; p, q\right) \\ & \prod_j \Gamma(t^2 v z_j^{\pm 1}; p, q) \Gamma\left(\frac{t^2}{w} z_j^{\pm 1}; p, q\right) \Gamma\left(\frac{wt^2}{v} z_j^{\pm 1}; p, q\right), \end{aligned}$$

and in the prefactor of the integrals,

$$\Gamma^{3n}(t^2; p, q) \rightarrow \Gamma^n(t^2 v; p, q) \Gamma^n\left(\frac{t^2}{w}; p, q\right) \Gamma^n\left(\frac{wt^2}{v}; p, q\right). \quad (\text{A.11})$$

B. The Representation Basis

The labels of the topological algebra as we have defined in (2.7) are (compact) continuous parameters $\alpha_i \in [0, 2\pi)$. We can “Fourier” transform to the discrete basis of irreducible $SU(2)$ representations. We denote by R_K the irreducible representation of $SU(2)$ of dimension $K+1$. The integrals over characters translate into sums over representations. The structure constants in the discrete basis are given by

$$\begin{aligned} C_{\alpha\beta\gamma} &= \sum_{K,L,M=0}^{\infty} \frac{\sin(K+1)\alpha}{\sin \alpha} \frac{\sin(L+1)\beta}{\sin \beta} \frac{\sin(M+1)\gamma}{\sin \gamma} \hat{C}_{KLM} \quad (\text{B.1}) \\ &= \sum_{K,L,M=0}^{\infty} \chi_K(\alpha) \chi_L(\beta) \chi_M(\gamma) \hat{C}_{KLM}, \end{aligned}$$

where $\chi_K(\alpha)$ is the character of R_K ,

$$\chi_K(\alpha) = \frac{\sin(K+1)\alpha}{\sin \alpha}. \quad (\text{B.2})$$

Similarly the metric in the discrete basis is given by

$$\eta^{\alpha\beta} = \sum_{K,L=0}^{\infty} \chi_K(\alpha) \chi_L(\beta) \hat{\eta}^{KL}. \quad (\text{B.3})$$

Further, we define the scalar product of characters³

$$\begin{aligned} \langle \chi_K \chi_M \rangle &= \frac{1}{2\pi i} \oint \frac{dz}{z} (1-z^2) \chi_K(z) \chi_M(z) \\ &= -\frac{1}{4\pi i} \oint \frac{dz}{z} \left(z - \frac{1}{z}\right)^2 \chi_K(z) \chi_M(z) = \int_0^{2\pi} d\theta \Delta(\theta) \chi_K(\theta) \chi_M(\theta) = \delta_{K,M}. \end{aligned} \quad (\text{B.4})$$

In the second equality we have introduced the measure (2.6) and used the fact that $\chi(z) = \chi(z^{-1})$. Thus we have

$$\sum_{K=0}^{\infty} \chi_K(\alpha) \chi_K(\beta) = \hat{\delta}(\alpha, \beta), \quad \int_0^{2\pi} d\theta \Delta(\theta) \hat{\delta}(\theta, \alpha) f(\theta) = f(\alpha), \quad (\text{B.5})$$

for any f obeying $f(\theta) = f(-\theta)$. Using (2.7) we can write

$$\hat{\eta}^{KL} = \eta^I \langle \chi^I \chi^K \chi^L \rangle, \quad \eta^I = \int d\alpha \Delta(\alpha) \eta^\alpha \chi_I(\alpha). \quad (\text{B.6})$$

Finally with the help of these definitions, we can rewrite (2.9) as

$$\mathcal{I} = \prod_{\{i,j,k\} \in \mathcal{V}} \hat{C}_{L_i L_j L_k} \prod_{\{m,n\} \in \mathcal{G}} \hat{\eta}^{L_m L_n}, \quad (\text{B.7})$$

where index contractions now indicate sums over the non-negative integers.

C. TQFT Algebra for $v = t$

For $v = t$ we can rewrite the algebra of the topological quantum field theory (2.7) in a more elegant way, removing the delta-functions by making use of identities obeyed by elliptic Beta integrals. This does not appear to be a preferred limit physically, except for the fact that the contribution to the index of the chiral superfield in the $\mathcal{N} = 2$ vector multiplet vanishes, see

³We have a slightly different convention for the characters and thus the expression of the scalar product differs from the one in [37].

(3.2). Our manipulations will be slightly formal since the limit $v = t$ of the formulae we will use is somewhat singular. We start by quoting the important identity

$$E^{(m=0)}(t_1, \dots, t_6) = \kappa \oint \frac{dz}{z} \frac{\prod_{k=1}^6 \Gamma(t_k z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} = \prod_{1 \leq j < k \leq 6} \Gamma(t_j t_k; p, q), \quad \prod_{k=1}^6 t_k = pq. \quad (\text{C.1})$$

This is a vast generalization to elliptic Gamma functions of that seminal object in string theory, the classic Beta integral of Euler,

$$B(\alpha, \beta) = \int_0^1 dt t^{\alpha-1} (1-t)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (\text{C.2})$$

which is recovered as a special limit, see *e.g.* [26]. Applying (C.1) we have

$$\kappa \oint \frac{dz}{z} \frac{\Gamma(\tau \sqrt{\nu} a^{\pm 1} b^{\pm 1} z^{\pm 1}) \Gamma(\frac{\tau}{\nu} z^{\pm 1} y^{\pm 1})}{\Gamma(z^{\pm 2})} = \Gamma\left(\frac{\tau^2}{\sqrt{\nu}} a^{\pm 1} b^{\pm 1} y^{\pm 1}\right) \Gamma(\tau^2 \nu a^{\pm 2}) \Gamma(\tau^2 \nu b^{\pm 2}) \Gamma\left(\frac{\tau^2}{\nu^2}\right) \Gamma(\tau^2 \nu)^2. \quad (\text{C.3})$$

For brevity we have omitted the p and q parameters in the Gamma functions. We assume $pq = \tau^6$. For these values of p and q , $\Gamma(\tau^3 z^{\pm 1}) = 1$. Now if we take $\nu = \tau$,

$$\kappa \oint \frac{dz}{z} \frac{\Gamma(\tau^{3/2} a^{\pm 1} b^{\pm 1} z^{\pm 1}) \Gamma(z^{\pm 1} y^{\pm 1})}{\Gamma(z^{\pm 2})} = \Gamma(\tau^{3/2} a^{\pm 1} b^{\pm 1} y^{\pm 1}) \Gamma(1). \quad (\text{C.4})$$

Strictly speaking the elliptic Beta integral formula (C.1) holds when $|t_k| < 1$ for all $k = 1 \dots 6$. For $\nu = \tau$ some of the t_k s in (C.3) saturate this bound. The elliptic Beta integral (C.3) is proportional to $\Gamma(\frac{\tau^2}{\nu^2}; p, q) \rightarrow \Gamma(1; p, q)$. Since the elliptic Gamma function has a simple pole when its argument approaches $z = 1$ (see (3.9)), (C.3) diverges in the limit. We will proceed by keeping formal factors of $\Gamma(1)$ in all the expressions. Thanks to (C.4), the expression

$$\frac{\Gamma(z^{\pm 1} y^{\pm 1})}{\Gamma(z^{\pm 2}) \Gamma(1)} \equiv \delta_y^z \quad (\text{C.5})$$

acts as a formal identity operator. All factors of $\Gamma(1)$ will cancel in the final expression for the index.

For $t = v$ we can write the building blocks of the topological algebra in the form summarized in Table 3. Contraction of indices is defined as

$$A^{\cdot\cdot a \cdot\cdot} B_{\cdot\cdot a \cdot\cdot} \rightarrow \kappa \oint \frac{da}{a} A^{\cdot\cdot a \cdot\cdot} B_{\cdot\cdot a \cdot\cdot}. \quad (\text{C.6})$$

We now proceed to perform a few sample calculations and consistency checks. We can raise an index of the structure constants to obtain

$$C_{abe} \eta^{ec} = \frac{\kappa}{\Gamma(1)} \oint \frac{de}{e} \Gamma(t^{\frac{3}{2}} a^{\pm 1} b^{\pm 1} e^{\pm 1}) \frac{\Gamma(e^{\pm 1} c^{\pm 1})}{\Gamma(e^{\pm 2}, c^{\pm 2})} = \frac{\Gamma(t^{\frac{3}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1})}{\Gamma(c^{\pm 2})} = C_{ab}^c. \quad (\text{C.7})$$



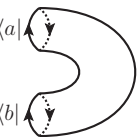
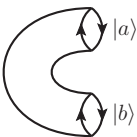
Symbol	Surface	Value	Symbol	Surface	Value
C_{abc}		$\Gamma(t^{\frac{3}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1})$	V^a		$\frac{1}{\Gamma(1)^2} \frac{\Gamma(t^{\pm \frac{3}{2}} a^{\pm 1})}{\Gamma(a^{\pm 2})}$
η^{ab}		$\frac{1}{\Gamma(1)} \frac{\Gamma(a^{\pm 1} b^{\pm 1})}{\Gamma(a^{\pm 2}, b^{\pm 2})}$	η_{ab}		$\frac{1}{\Gamma(1)} \Gamma(a^{\pm 1} b^{\pm 1})$

Table 3: The basic building blocks of the topological algebra in the $v = t$ case.

In particular we see that the index (3.17) is finite and is simply given by $C_{ab}{}^c C_{cde}$. The “vacuum state” $|V\rangle \equiv V^a |a\rangle$ satisfies by definition (see *e.g.* [33]) $C_{abc} V^c = \eta_{ab}$, as illustrated in Figure 8. This determines V^a to be the expression in Table 3,

$$C_{abc} V^c = \frac{\kappa}{\Gamma(1)^2} \oint \frac{dz}{z} \Gamma(t^{\frac{3}{2}} a^{\pm 1} b^{\pm 1} z^{\pm 1}) \frac{\Gamma(t^{\pm \frac{3}{2}} z^{\pm 1})}{\Gamma(z^{\pm 2})} = \frac{1}{\Gamma(1)} \Gamma(a^{\pm 1} b^{\pm 1}) = \eta_{ab}. \quad (\text{C.8})$$

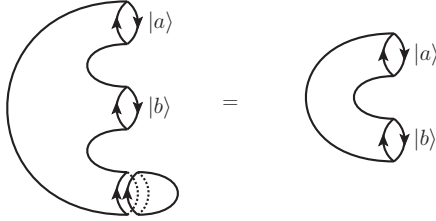


Figure 8: Constructing the metric by capping off the trivalent vertex.

Further, we can check that η_{ab} and η^{ab} in Table 3 are one the inverse of the other,

$$\eta^{ae} \eta_{ec} = \frac{\kappa}{\Gamma(1)^2} \oint \frac{de}{e} \frac{\Gamma(a^{\pm 1} e^{\pm 1})}{\Gamma(a^{\pm 2}, e^{\pm 2})} \Gamma(e^{\pm 1} c^{\pm 1}) = \frac{1}{\Gamma(1)} \frac{\Gamma(a^{\pm 1} c^{\pm 1})}{\Gamma(a^{\pm 2})} = \delta_c^a. \quad (\text{C.9})$$

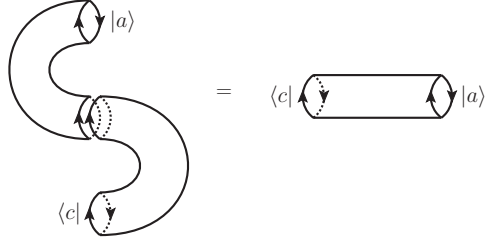


Figure 9: Topological interpretation of the property $\eta^{ce} \eta_{ea} = \delta_a^c$.

As a consistency check one can verify in examples that δ_b^a is indeed an identity. For instance

$$\delta_a^z C_{zbc} = \frac{\kappa}{\Gamma(1)} \oint \frac{dz}{z} \frac{\Gamma(a^{\pm 1} z^{\pm 1})}{\Gamma(z^{\pm 2})} \Gamma(t^{\frac{3}{2}} z^{\pm 1} b^{\pm 1} c^{\pm 1}) = \Gamma(t^{\frac{3}{2}} a^{\pm 1} b^{\pm 1} c^{\pm 1}) = C_{abc}, \quad (\text{C.10})$$

as illustrated in Figure 10. For completeness we can also compute the sphere and the torus

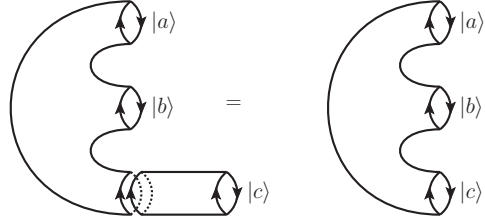


Figure 10: The consistency requirement $\delta_c^z C_{abz} = C_{abc}$.

partition functions. (These partition functions do not appear in any index computation of a 4d superconformal theory so their physical interpretation is unclear.)

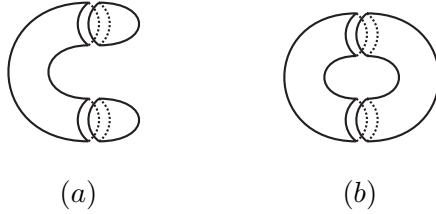


Figure 11: The sphere **(a)** and the torus **(b)** partition functions.

The sphere partition function is given by

$$\begin{aligned} V^c V^e \eta_{ce} &= \frac{\kappa^2}{\Gamma(1)^5} \oint \frac{de}{e} \oint \frac{dc}{c} \frac{\Gamma(c^{\pm 1} e^{\pm 1}) \Gamma(t^{\pm 3/2} c^{\pm 1}) \Gamma(t^{\pm 3/2} e^{\pm 1})}{\Gamma(c^{\pm 2}) \Gamma(e^{\pm 2})} = \\ &= \frac{\kappa}{\Gamma(1)^4} \oint \frac{de}{e} \frac{\Gamma(t^{\pm 3/2} e^{\pm 1})^2}{\Gamma(e^{\pm 2})} = \Gamma(t^{-3}) \frac{1}{\Gamma(1)}. \end{aligned} \quad (\text{C.11})$$

The torus partition function is given by

$$\eta_{ab} \eta^{ab} = \frac{\kappa}{\Gamma(1)} \oint \frac{da}{a} \frac{\Gamma(a^{\pm 1} a^{\pm 1})}{\Gamma(a^{\pm 2})} = \kappa \Gamma(1) \oint \frac{da}{a} = 2\pi i \kappa \Gamma(1). \quad (\text{C.12})$$

Since $\Gamma(1) = \infty$ the sphere partition function vanishes and the torus partition function diverges.

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